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# COMPARABILITY OF THE MOTION OF GRAVITATING SYSTEMS WITH RESPECT TO ITS RECURRENCE IN TIME*

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The sufficient condition of comparability of the motion of gravitating systems recurrent in time, and of its various characteristics treated as functions of time, is given.

**1. Formulation of the problem.** Consider a mechanical system of $n$ bodies $(G_n)$ consisting of material points $O_1, O_2, \ldots, O_n$ with masses $M_1, M_2, \ldots, M_n$, mutually attracting in accordance with Newton's Law. Let $x_i, y_i, z_i$ be the position coordinates, $u_i, v_i, w_i$ the velocity components of the body $O_i$ in the barycentric frame of reference $Oxyz$, and let

$$U = \gamma \sum_{1\leqslant i<j\leqslant n} \frac{M_i M_j}{r_{ij}}$$

be the force function of the system $G_n$ where $\gamma$ is the gravitational constant and $r_{ij}$ the distance between the bodies $O_i, O_j$ $(i \neq j = 1, 2, \ldots, n)$. The motion of the system $G_n$ is described by the solution of the differential equations /1/

$$\begin{gathered} x_i^{\cdot} = u_i, \quad y_i^{\cdot} = v_i, \quad z_i^{\cdot} = w_i \\ M_i u_i^{\cdot} = \partial U/\partial x_i, \quad M_i v_i^{\cdot} = \partial U/\partial y_i, \quad M_i w_i^{\cdot} = \partial U/\partial z_i \end{gathered} \tag{1.1}$$

satisfying the initial conditions for the form

$$\begin{gathered} x_i(0) = x_{i0}, \quad y_i(0) = y_{i0}, \quad z_i(0) = z_{i0} \\ u_i(0) = u_{i0}, \quad v_i(0) = v_{i0}, \quad w_i(0) = w_{i0} \end{gathered} \tag{1.2}$$

Let

$$\begin{gathered} q = \{x_{10}, y_{10}, \ldots, z_{n0}, u_{10}, v_{10}, \ldots, w_{n0}\} \\ g(t, q) = \{x_1(t, q), y_1(t, q), \ldots, z_n(t, q), u_1(t, q), v_1(t, q), \ldots, \\ w_n(t, q)\} \end{gathered}$$

be the 6-dimensional vector and vector function representing, respectively, the initial state of the motion of $G_n$ and

$$T[g(t, q)] = \frac{1}{2} \sum_{i=1}^{h} M_i [u_i^2(t, q) + v_i^2(t, q) + w_i^2(t, q)]$$

its kinetic energy along the trajectory of motion $g(t, q)$, where $t$ is the time.

If the motion $g(t, q)$ of system $G_n$ is Poisson stable recurrently and periodically, or almost periodically, then the kinetic energy $T[g(t, q)]$ of the system, regarded as a function of time, possesses every property of recurrence of the motion with time described above. We pose the problem of inverting this assertions: if the kinetic energy $T[g(t, q)]$ of the system, regarded as a function of time, is Poisson stable, recurrent, and periodic or almost periodic, what additional conditions are required for the motion $g(t, q)$ to possess the time recurrence properties listed above?

**2. Dynamic systems in metric spaces.** Let $M$ be a complete metric space with the metric $\rho$, and let $g(t, \cdot)$ denote a one-parameter family of mappings of $M$ onto iteself, defined for all real values of the parameter $t \in R_1$ and the elements $q \in M$, where $t$ is time.

The terminology, concepts and definitions used are those of /2-4/.

We denote by $g(R_1, q)$, $g(R_1^+, q)$ and $g(R_1^-, q)$ the trajectory and the positive and negative half-trajectories of the motion $g(t, q)$. We have $R_1^- = \{t \leqslant 0\}$, $R_1^+ = \{t \geqslant 0\}$ and the quantity $q$ is fixed.

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We shall call negatively (positively) stable and Lagrange stable motions, $\mathbf{L}^-$ ($\mathbf{L}^+$) -stable and $\mathbf{L}$-stable for brevity. Similarly, we shall call negatively (positively) stable and Poisson stable motions, $\mathbf{P}^-$ ($\mathbf{P}^+$)-stable and $\mathbf{P}$-stable respectively.

Note that if the motion $g(t, q)$ is $\mathbf{P}^-$ ($\mathbf{P}^+$)-stable, then the following inclusion holds:

$$g(R_1, q) \subset \overline{g(R_1^-, q)} \quad (g(R_1, q) \subset \overline{g(R_1^+, q)}) \tag{2.1}$$

*Proposition 2.1.* If the motion $g(t, q)$ is $\mathbf{L}^-$-stable and $\mathbf{P}^-$-stable, then it is $\mathbf{L}$-stable.

Indeed, since the motion $g(t, q)$ is $\mathbf{P}^-$-stable, the inclusion (2.1) holds and the $\mathbf{L}^-$-stability of the motion implies that the set $\overline{g(R_1^-, q)}$ is compact; therefore the set $\overline{g(R_1, q)}$ is also compact.

Let $\varepsilon$ be an arbitrary, positive real number, $g(a \leqslant t \leqslant b, q) = \{g(t, q) \in M \mid t \in [a, b]\}$ the arc of the trajectory $g(R_1, q)$ of time length $(b-a)$, and $B(g(a \leqslant t \leqslant b, q), \varepsilon) = \{p \in M \mid \rho[g(a \leqslant t \leqslant b, q), p] < \varepsilon\}$ the open $\varepsilon$-neighbourhood of the arc $g(a \leqslant t \leqslant b, q)$.

*Proposition 2.2.* The motion $g(t, q)$ is periodic if and only if the instants of time $t_1$ and $t_2$ exist such that $t_1 \neq t_2$ and $g(t_1, q) = g(t_2, q)$.

The necessity is obvious. Let $t_2 > t_1$, $t_2 - t_1 = s$ and $g(t_1, q) = g(t_2, q)$. Then by virtue of the group theoretic property of the mapping we have a chain of equalities: $g(-t_1, g(t_1, q)) = g(t_1 - t_1, q) = g(-t_1, g(t_2, q))$, $q = g(t_2 - t_1, q)$, $g(t, q) = g(t + s, q)$. Consequently $g(t, q)$ is a periodic motion and $s > 0$ is its period ($s$ is the optionally smallest period).

*Proposition 2.3.* If $g(t, \cdot)$ is a dynamic system defined in a complete metric space $M$, then three types of motion exist:

1) rest, i.e. $\forall t \in R_1$, $g(t, q) = q$;

2) periodic motion with the minimum positive period $\tau$, i.e. $g(t + \tau, q) = g(t, q)$ and $g(t_1, q) \neq g(t_2, q)$ for all $t_1$ and $t_2$ such that $0 \leqslant t_1 < t_2 < \tau$;

3) non-periodic motion, i.e. the inequality $g(t_1, q) \neq g(t_2, q)$ holds for all $t_1, t_2 \in R_1$ such that $t_1 \neq t_2$.

Let us assume that $g(t, q)$ does not belong to the third type of motions. Then $t_1 \neq t_2$, exists such that $g(t_1, q) = g(t_2, q)$. According to the previous proposition the motion $g(t, q)$ is periodic, therefore it either has the minimum positive period (i.e. it belongs to the second type of motions), or for all $t \in R_1$ $g(t, q) = q$, i.e. $g(t, q)$ belongs to the first type of motions. The proposition is proved.

Let $C(R_1, M)$ denote the set of all continuous functions $f(t)$ with the domain of definition $R_1$, and values belonging to the space $M$.

The real function $\rho_M$, which puts a real non-negative number in correspondence with every pair of functions $u(t)$, $v(t) \in C(R_1, M)$ according to the formula

$$\rho_M(u, v) = \sup_{l>0} \min \{\sup_{|t| \leqslant l} \rho[u(t), v(t)], 1/l\} \tag{2.2}$$

defines the metric in the space $C(R_1, M)$ [4]. Here the space $C(R_1, M)$ with the metric (2.2) is complete, and conditions $\rho_M(u, v) < \varepsilon$, $\rho_M(u, v) = \varepsilon$, $\rho_M(u, v) > \varepsilon$ hold for all $u, v \in C(R_1, M)$ and $\varepsilon > 0$, if and only if the following conditions hold:

$$\sup_{|t| \leqslant 1/\varepsilon} \rho[u(t), v(t)] < \varepsilon, \quad \sup_{|t| = 1/\varepsilon} \rho[u(t), v(t)] = \varepsilon$$

$$\sup_{|t| \geqslant 1/\varepsilon} \rho[u(t), v(t)] > \varepsilon$$

respectively.

Similarly we introduce, in the space $C(R_1, M)$ the dynamic system $g(s, \cdot)$ as a one-parameter family of mappings $C(R_1, M)$ into themselves. Here the motion $g(s, f(t))$ is defined as the displacements $f(t + s)$ in the function $f(t)$ where $f(t) \in C(R_1, M)$ is a fixed function. The trajectory of the motion $g(s, f(t))$ is represented by the set $g(R_1, f(t)) = \{f(t + s) \mid s \in R_1\}$ of all displacements of the function $f(t)$. The positive and negative half-trajectories of the motion $g(s, f(t))$ are obtained analogously.

*Birkhof's theorem /4/.* The function $f(t) \in C(R_1, M)$ is recurrent if and only if it is $\mathbf{L}$-stable and minimal.

We denote by $W(\alpha, f)$ and $W(\omega, f)$ the sets of all characteristic $\alpha$- and $\omega$-sequences of the functions $f(t) \in C(R_1, M)$ and $W(\lambda, f) = W(\alpha, f) \cup W(\omega, f)$.

Let $K$ be the complete metric space with the metric $\cdot\, \sigma$, and let $C(R_1, K)$ be the space of all continuous functions with values in $K$ and metric $\sigma_K$.

*Definition 2.1.* The function $u(t) \in C(R_1, M)$ will be called negatively (positively) comparable with respect to time recurrence (in short: $R^-$ ($R^+$) -comparable) with the function $v(t) \in C(R_1, K)$, provided the following inclusion holds:

$$W(\alpha, v) \subset W(\alpha, u) \quad (W(\omega, v) \subset W(\omega, u)) \tag{2.3}$$

If $W(\lambda, v) \subset W(\alpha, u)$, then the function $u(t)$ will be comparable with respect to time recurrence (in short: $R$-comparable) with the function $v(t)$.

*Definition 2.2.* We shall call the functions $u(t) \in C(R_1, M)$, $v(t) \in C(R_1, K)$ negatively (positively) isochronous with respect to time recurrence (in short: $R^-(R^+)$-isochronous), if the following relation holds:

$$W(\alpha, u) = W(\alpha, v) \quad (W(\omega, u) = W(\omega, v)) \tag{2.4}$$

If on the other hand $W(\lambda, u) = W(\lambda, v)$, then the functions $u(t)$, $v(t)$ will be called isochronous with respect to time recurrence (in short: $R$-isochronous).

*Proposition 2.4.* The function $u(t) \in C(R_1, M)$ is $R^-(R^+)$-comparable with the function $v(t) \in C(R_1, K)$ if and only if the following condition A holds: that a subsequence $\{t_{n_k}\} \in W(\alpha, u)$ $(\in W(\omega, u))$ can be extracted from every sequence $\{t_m\} \in W(\alpha, u)$ $(\in W(\omega, u))$.

The necessity of the condition follows directly from the inclusion (2.3). Conversely, let condition A hold, but $u(t)$ be not $R^-$-comparable with the function $v(t)$. Then the function $v(t)$ will have a characteristic $\alpha$-sequence $\{t_m\} \in W(\alpha, v)$, such that $\{t_m\} \notin W(\alpha, u)$. Therefore, we can extract from $\{t_m\}$ a sequence $\{t_{m_k}\}$, such that no subsequence belonging to it appears in the set $W(\alpha, u)$. This contradicts condition A, thus proving the sufficiency of proposition 2.4.

*Proposition 2.5.* If the function $u(t) \in C(R_1, M)$ is $R^-(R^+)$-comparable with periodic function $v(t) \in C(R_1, K)$, then $u(t)$ is periodic and any period of the function $v(t)$ is also a period of $u(t)$.

Let $s$ denote any non-zero period of the function $v(t)$, i.e. $v(t+s) = v(t)$, $\forall t \in R_1$. If $s > 0$, then the sequence $\{-ks\}_{k \in N}$ belongs to the set $W(\alpha, v)$. If on the other hand $s < 0$, then the sequence $\{ks\}_{k \in N} \in W(\alpha, v)$. Therefore, depending on the sign of the period $s$, one of the sequence $\{\pm ks\}_{k \in N}$ will belong to the set $W(\alpha, v)$ and thus also to the set $W(\alpha, u)$. We shall assume, to be specific that $\{ks = t_k\}_{k \in N} \in W(\alpha, u)$. Consequently $\lim \rho_M [u(t_k + t), u(t)] = 0$ as $k \to \infty$. and we have

$$u(t) = \lim_{k \to \infty} u(t + t_k) = \lim_{k \to \infty} u(t + t_k + t_{k+1} - t_k) = \lim_{k \to \infty} u[t + t_k + (k+1)s - ks] = \lim_{k \to \infty} u(t + s + t_k) = u(t+s)$$

i.e. $u(t) = u(t+s)$. This means that $s$ is a period of the function $u(t)$.

*Proposition 2.6.* The function $u(t) \in C(R_1, M)$ is $R$-comparable with periodic function $v(t) \in C(R_1, K)$ if and only if $u(t)$ is $R^-(R^+)$-comparable with $v(t)$.

The necessity is obvious. Let $u(t)$ be $R^-$-comparable with periodic function $v(t)$. According to proposition 2.4 the function $u(t)$ is periodic and every period of $v(t)$ is also a period of $u(t)$. Let $s$ be the smallest positive period of the function $u(t)$, and let $\{t_k\}$ be an arbitrary sequence belonging to the set $W(\omega, v)$. We can find, for every natural number $k$, an integer $s_k$ is such a manner that $s_k s \leqslant t_k < (s_k + 1)s$. Here the sequence $\{l_k\}_{k \in N}$, where $l_k = t_k - s_k s$, is bounded, and we can extract from it a convergent subsequence $\{l_{k_i}\}$:

$$\lim_{i \to \infty} l_{k_i} = l, \quad 0 \leqslant l \leqslant s$$

Then

$$v(t+l) = \lim_{i \to \infty} v(t + l_{k_i}) = \lim_{i \to \infty} v(t + t_{k_i} - s_{k_i} s) = \lim_{i \to \infty} v(t + t_{k_i}) = v(t)$$

This means that $l$ is a period of $v(t)$. Therefore we have either $l = 0$, or $l = s$. In each of these cases we have

$$\lim_{i \to \infty} u(t + t_{k_i}) = \lim_{i \to \infty} u(t + l_{k_i} + s_{k_i} s) = \lim_{i \to \infty} u(t + l_{k_i}) = u(t+l) = u(t)$$

Thus $\{t_{k_i}\} \in W(\omega, u)$, i.e. condition A holds. Then according to (2.5) $u(t)$ is $R^+$-comparable with $v(t)$.

From Propositions 2.5 and 2.6 and Theorem 2.1.1 of /4/, we have

*Proposition 2.7.* The function $u(t) \in C(R_1, M)$ is $R^-(R^+)$-comparable with periodic function $v(t) \in C(R_1, K)$ if and only if $u(t)$ is periodic and every period of $v(t)$ is also a period of $u(t)$.

*Proposition 2.8.* The necessary and sufficient condition for the function $u(t) \in C(R_1, M)$ to be $R^-(R^+)$-comparable with the function $v(t) \in C(R_1, K)$, is that a continuous mapping $\varphi$ of the set $g(R_1^-, u(t))$ onto the set $g(R_1^-, v(t))$ exists for all $s \in R_1^-$, the mapping satisfying the relation

$$\varphi [v(t+s)] = u(t+s), \quad t \in R_1 \tag{2.5}$$

The proof is analogous to that of Theorem 2.1.2 of /4/, the latter expressing the comparability of $u(t)$ with the function $v(t)$.

Let $M$ be a Banach space $\rho(u, v) = \| u - v \|$, and $f(t) \in C(R_1, M)$. We have the following two propositions /4/.

*Proposition 2.9.* If the derivative $u^{\cdot}(t) \in C(R_1, M)$ of the function $u(t) \in C(R_1, M)$ is uniformly continuous on $R_1^-(R_1^+)$, then it is $R^-(R^+)$-comparable with $u(t)$.

*Proposition 2.10.* If $f(t) \in C(R_1, M)$ and has a negatively (positively) compact primitive $u(t) \in C(R_1, M)$, then $u(t)$ is $R^-(R^+)$-comparable with the function $f(t)$.

**3. $R^-$-isochronism of the kinetic energy and motion of the $L^-$-stable system $G_n$.** Let

$$I[g(t,q)] = \sum_{i=1}^{n} M_i [x_i^2(t,q) + y_i^2(t,q) + z_i^2(t,q)]$$

be the polar moment of inertia of the system $G_n$, and

$$I^{\cdot}[g(t,q)] = 2 \sum_{i=1}^{n} M_i [x_i(t,q) u_i(t,q) + y_i(t,q) v_i(t,q) + z_i(t,q) w_i(t,q)] \tag{3.1}$$

$$I^{\cdot\cdot}[g(t,q)] = 2 \{T[g(t,q) + h\} \tag{3.2}$$

be the first and second derivative with respect to time t, of the function $I[g(t,q)]$, where $h$ is the energy constant.

Equation (3.2) is called the Lagrange-Jacobi equation /1/.

Note that the motion $g(t,q)$ is generated by the dynamic system if and only if $T[g(t,q)] \in C(R_1, R_1^+)$.

We shall henceforth assume that the variables in $g(t,q)$, are dimensionless, and put $Q[g(t,q)] = I[g(t,q)] + 2T[g(t,q)]$. We have $Q[g(t,q)] = \| g(t,q) \|^2$ for every fixed $t \in R_1$. This yields.

*Proposition 3.1.* The motion $g(t,q)$ of the system $G_n$ is $L^-$-stable if and only if the function $Q[g(t,q)] = Q(t)$, $Q(t) \in C(R_1, R_1^+)$. is $L^-$-stable.

Since the function $I[g(t,q)]$, $T[g(t,q)]$ is non-negative, we have

*Proposition 3.2.* The function $Q[g(t,q)]$ is $L^-$-stable if and only if the functions $I[g(t,q)]$, $T[g(t,q)]$ are $L^-$-stable.

*Proposition 3.3.* If the motion $g(t,q)$ of the system $G_n$ is $L^-$-stable, then the functions $I[g(t,q)] = I(t)$, $I^{\cdot}[g(t,q)] = I^{\cdot}(t)$, $I^{\cdot\cdot}[g(t,q)] = I^{\cdot\cdot}(t)$, $T[g(t,q)] = T(t)$ are $R^-$-isochronous.

The conditions of the proposition imply that the motion $g(t,q)$ is $L^-$-stable. Then, by virtue of Proposition 3.1 the function $Q[g(t,q)]$ will be $L^-$-stable, while by virtue of Proposition 3.2, every function $I[g(t,q)]$, $T[g(t,q)]$ will be $L^-$-stable. Therefore the following inequalities hold:

$$\sup_{t \in R_1^-} |I(t)| < \infty, \quad \sup_{t \in R_1^-} |I^{\cdot}(t)| < \infty, \quad \sup_{t \in R_1^-} |I^{\cdot\cdot}(t)| < \infty$$

from which the uniform continuity in $R_1^-$ of the functions $I(t)$, $I^{\cdot}(t)$, $I^{\cdot\cdot}(t)$ and $T(t)$ follows. This in turn yields, according to Proposition 2.6. the $R^-$-comparability of the functions $I^{\cdot}(t)$ with $I(t)$, $I^{\cdot\cdot}(t)$ with $I^{\cdot}(t)$, and $I^{\cdot\cdot}(t)$ with $T(t)$. Since the $L^-$-stability implies the negative compactness, it follows by virtue of Proposition 2.10, that the function $T(t)$ will be comparable with the function $I^{\cdot\cdot}(t)$, $I^{\cdot}(t)$ with $I^{\cdot\cdot}(t)$ and $I(t)$ with $I^{\cdot}(t)$. Consequently, the functions listed above in Proposition 3.3 are $R^-$-isochronous with respect to each other.

*Proposition 3.4.* If the motion $g(t,q)$ of the system $G_n$ is $L^-$-stable, then the functions $Q(t)$ and $T(t)$ are $R^-$-isochronous.

Let $\{t_k\} \in W(\alpha, T)$. According to the proposition 3.3, $\{t_k\} \in W(\alpha, I)$. These inclusions indicate that

$$\lim_{k\to\infty} T(t+t_k) = T(t), \quad \lim_{k\to\infty} I(t+t_k) = I(t)$$

Then

$$\lim_{k\to\infty} Q(t+t_k) = \lim_{k\to\infty} [I(t+t_k) + 2T(t+t_k)] = \lim_{k\to\infty} I(t+t_k) + 2\lim_{k\to\infty} T(t+t_k) = I(t) + 2T(t) = Q(t)$$

i.e. $\{t_k\} \in W(\alpha, Q)$, and according to Proposition 2.4 the function $Q(t)$ is $R^-$-comparable with $T(t)$.

Conversely, let $\{t_k\} \in W(\alpha, Q)$, i.e.

$$\lim_{k\to\infty} Q(t+t_k) = Q(t)$$

Then

$$\lim_{k\to\infty} [Q(t+t_k)+2h] = Q(t)+2h$$

Since $Q(t)+2h = I(t)+2[T(t)+h]$, we obtain by virtue of (3.2), the differential equation $I^{\cdot\cdot}(t)+I(t) = Q(t)+2h$. Recalling that in addition to $g(t, q)$ the expression $g(t+s, q)$ is also a solution of problem (1.1), (1.2) for every fixed $s \in R_1$, we obtain

$$I^{\cdot\cdot}(t+t_k)+I(t+t_k) = Q(t+t_k)+2h$$

The equation has a particular solution of the form

$$I(t+t_k) = \int_0^t \sin(t-s)[Q(t_k+s)+2h]\,ds$$

from which we find, taking the inclusion $t_k \in W(\alpha, Q)$ into account, that

$$\lim_{k\to\infty} I(t+t_k) = \int_0^t \sin(t-s)[Q(s)+2h]\,ds = I(t)$$

i.e. $\{t_k\} \in W(\alpha, I)$ and $I(t)$ is $R^-$-comparable with the function $Q(t)$.

According to Proposition 3.3, the functions $I(t)$ and $T(t)$ are $R^-$-isochronous. Therefore $\{t_k\} \in W(\alpha, I)$ with $T(t)$ is $R^-$-comprable with $Q(t)$. Then the mutually $R^-$-comparable functions $Q(t)$ and $T(t)$ are $R^-$-isochronous.

The following proposition is a corollary to Propositions 3.3 and 3.4.

*Proposition 3.5.* If the motion of the system $G_n$ is $\mathbf{L}^-$-stable, then the functions $I(t)$, $I^{\cdot}(t)$, $I^{\cdot\cdot}(t)$, $T(t)$ and $Q(t)$ are $R^-$-isochronous.

Let us denote by $F[g(t, q)]$, $F[Q(t)]$ the sets of all displacements of the functions $g(t, q)$ and $Q(t)$, and by $\varphi$ the mapping of the set $F[g(t, q)]$ onto the set $F[Q(t)]$, defined for all fixed $s \in R_1$ by the relation $\varphi[g(t+s, q)] = Q[g(t+s, q)] = Q(t+s)$. It is clear here that $\varphi[g(t+s, q)] = \| g(t+s, q)\|^2$ for every fixed $t \in R_1$ and for all $s \in R_1$. Therefore the following proposition holds.

*Proposition 3.6.* The mapping $\varphi$ is single-valued, continuous and closed.

We note that the continuous and closed mappings belong to the class of so-called factorial mappings /5/.

*Proposition 3.7.* The motion $g(t, q)$ is non-periodic and the mapping $\varphi$ is in 1:1 correspondence if and only if the function $Q(t)$ is non-periodic.

*Necessity.* Let the motion $g(t, q)$ be non-periodic and the mapping $\varphi$ be in 1:1 correspondence. Then for all $t_1 \neq t_2$ we have

$$g(t+t_1, q),\ g(t+t_2, q) \in F[g(t, q)],\ g(t+t_1, q) \neq g(t+t_2, q)$$

and the mapping $\varphi$ places in 1:1 correspondence such elements $Q(t+t_1)$ and $Q(t+t_2)$, that $Q(t+t_1) \neq Q(t+t_2)$. Therefore we have, in particular, $Q(t) \neq Q(t+s)$, $\forall s \in R_1$. Consequently no real number $s$ is a period of the function $Q(t)$. In other words, the function $Q(t)$ is non-periodic.

*Sufficiency.* Let the function $Q(t)$ be non-periodic, i.e. $\forall s \in R_1, Q(t) = Q(t+s)$, where $s \neq 0$. Then $g(t, q) \neq g(t+s, q)$ even more so, and $g(t, q)$ is non-periodic. We shall assume that under these conditions the mapping $\varphi$ is not in 1:1 correspondence. Then the function $Q(t+t_1)$ has, for some $t_1 \in R_1$, at least two inverse images $g(t+t_1, q)$, $g(t+t_2, q)$, such that $\| g(t+t_1, q)\|^2 = \| g(t+t_2, q)\|^2 = Q(t+t_1)$ and $t_2 - t_1 > 0$. Therefore $\| g(t, q)\|^2 = \| g(t+\tau, q)\|^2 = Q(t+\tau) = Q(t)$ for all $t \in R_1$, where $\tau = t_2 - t_1 > 0$. This contradicts the condition that $Q(t)$ is a non-periodic function, and the contradiction proves the sufficiency of the proposition.

Knowing that every factorial mapping in 1:1 correspondence is a homeomorphism, we use Propositions 2.8 and 3.7 to obtain

*Proposition 3.8.* If the motion $g(t, q)$ of the system $G_n$ is $\mathbf{L}^-$-stable and the function $Q(t)$ is non-periodic, then the motion $g(t, q)$ and kinetic energy $T[g(t, q)]$ along the trajectory of this motion are $R^-$-isochronous as functions of time.

## 4. Certain properties of the L⁻-stable motion of a system with recoverable kinetic energy.

Proposition 3.8 implies that if the motion $g(t, q)$ of the system $G_n$ is $\mathbf{L}^-$-stable and the kinetic energy $T[g(t, q)]$ non-periodic, then the motion $g(t, q)$ is non-perodic. Taking this into account we shall first consider the case of the non-periodic function $T[g(t, q)]$.

*Proposition 4.1.* If the motion $g(t, q)$ of system $G_n$ is $\mathbf{L}^-$-stable and the kinetic energy

$T[g(t, q)]$ is $\mathbf{P}^-$-stable, then the motion $g(t, q)$ of the system is $\mathbf{L}$-stable and $R$-isochronous with the function $T[g(t, q)]$.

Indeed, the function $Q[g(t, q)]$ and motion $g(t, q)$ are $\mathbf{L}^-$-stable, and together with the function $T[g(t, q)]$, also $\mathbf{P}^-$-stable. According to Proposition 2.1, since the motion $g(t, q)$ is simultaneously $\mathbf{L}^-$- and $\mathbf{P}^-$-stable, it is also $\mathbf{L}$-stable. Then, according to Proposition 3.8 it is $R$-isochronous with the function $T[g(t, q)]$.

Proposition 4.1 has a converse, i.e. the motion $g(t, q)$ of the system $G_n$ is $\mathbf{L}$-stable and $\mathbf{P}$-stable if and only if it is $\mathbf{L}^-$-stable and the kinetic energy $T[g(t, q)]$ is $\mathbf{P}$-stable.

*Proposition 4.2.* If the motion $g(t, q)$ of the system $G_n$ is $\mathbf{L}^-$-stable and the kinetic energy $T[g(t, q)]$ is minimal, the motion $g(t, q)$ of the system is recurrent.

According to Proposition 3.2 the function $T[g(t, q)]$ is $\mathbf{L}^-$-stable, or the motion $g(t, q)$ is $\mathbf{L}^-$-stable. Therefore, the function $T[g(t, q)]$ is $\mathbf{L}^-$-stable. According to the *Birkhof theorem* the $\mathbf{L}$-stable and minimal function $T[g(t, q)]$ is recurrent, and therefore even more $\mathbf{P}$-stable. Consequently, the motion $g(t, q)$ is $\mathbf{L}$-stable. Then, in accordance with the theorem on the integrals of recurrent functions, the function $Q[g(t, q)]$ is also recurrent and $R$-isochronous with the motion $g(t, q)$. Therefore a homeomorphism $\varphi$ exists between the sets $F[g(t, q)]$ and $F[Q(t)]$. In particular, the inverse mapping $\varphi^{-1}$ is continuous at the point $Q(t) \in F[Q(t)]$, i.e. for every $\varepsilon > 0$ a number $\delta(\varepsilon, Q(t)) > \varepsilon$, can be found such, that

$$\sup_{|t| \leqslant 1/\varepsilon} \| g(t+s, q) - g(t, q) \| \leqslant \varepsilon$$

so long as

$$\sup_{|t| \leqslant 1/\varepsilon} | Q(t+s) - Q(t) | \leqslant \delta$$

Let $\varepsilon$ be any fixed number. Since $\varphi^{-1}$ is continuous, we can find the number $\delta(\varepsilon, Q(t))$ at a point belonging to $Q(t)$. Since $Q(t)$ is a recurrent function, it follows that for $\delta$ thus obtained the set

$$R_\delta(Q) = \{ s \in R_1 \mid \sup_{|t| \leqslant 1/\delta} | Q(t+s) - Q(t) | \leqslant \delta \}$$

is relatively dense in $R_1$. Therefore, the set

$$R_\varepsilon(g) = \{ s \in R_1 \mid \sup_{|t| \leqslant 1/\varepsilon} \| g(t+s, q) - g(t, q) \| \leqslant \varepsilon \}$$

containing the set $R_\delta(Q)$, is relatively dense in $R_1$. Thus the motion $g(t, q)$ is $\mathbf{L}$-stable and the set $R_\varepsilon(g)$ is relatively dense in $R_1$ for any $\varepsilon > 0$. By virtue of the criterion of recurrent motions /2, 3/, the motion $g(t, q)$ is recurrent.

Proposition 4.2 also has a converse, and the following proposition holds: the motion $g(t, q)$ of the system $G_n$ is recurrent if and only if it is $\mathbf{L}^-$-stable and the kinetic energy is minimal.

Using the Bochner theorem on almost periodic motions /3/ we can prove, in exactly the same way.

*Proposition 4.3.* If the motion $g(t, q)$ of the system $G_n$ is $\mathbf{L}^-$-stable and the kinetic energy $T[g(t, q)]$ almost periodic, then the motion $g(t, q)$ of the system is almost periodic.

Proposition 4.3 also holds in a more general form: the motion $g(t, q)$ of the system $G_n$ is almost periodic if and only if it is $\mathbf{L}^-$-stable and the kinetic energy $T[g(t, q)]$ is almost periodic.

*Proposition 4.4.* If the motion $g(t, q)$ of the system $G_n$ is $\mathbf{L}^-$-stable and the kinetic energy $T[g(t, q)]$ is periodic, then the motion $g(t, q)$ of the system is periodic.

Let the motion $g(t, q)$ be $\mathbf{L}^-$-stable and the kinetic energy $T[g(t, q)]$ periodic. According to Propositions 3.2 and 3.5, the function $Q[g(t, q)]$ is periodic and the motion $g(t, q)$ $\mathbf{L}$-stable. Here either every real number $s \in R_1$ is a period of the function $Q[g(t, q)]$, or the function $Q[g(t, q)]$ has a minimum positive real period $\tau$. In the first of these cases the function $Q[g(t, q)]$ is constant and its total time derivative is equal to zero by virtue of system (1.1). Therefore, the motion $g(t, q)$ is periodic and Proposition 4.4 is thus proved.

Let us assume that $\tau \in R_1$ is the minimum period of the function $Q[g(t, q)]$. Then the set

$$F_k(g) = \{ g(t+s, q) \mid s \in [(k-1)\tau, k\tau) \}$$

will map onto the set $F_k(Q) = \{ Q[g(t+s, q)] \mid s \in [(k-1)\tau, k\tau) \}$ for any natural $k$, in 1:1 correspondence.

Indeed, if it is not so, the functions $g(t+s_1, q)$, $g(t+s_2, q)$ will have the same mapping for certain $s_1, s_2 \in [(k-1)\tau, k\tau)$, i.e. $Q[g(t+s_1, q)] = Q[g(t+s_2, q)]$ and the number $s = |s_1 - s_2| < \tau$ will be a period of the function $Q[g(t, q)]$. This contradicts the condition of the choice of $\tau$. Therefore the mapping $\varphi$ can be additionally defined so that it becomes a homemorphism. Then the function $Q[g(t, q)]$ and the motion $g(t, q)$ will be isochronous. According to Proposition 2.7 the motion $g(t, q)$ is periodic.

Like the previous propositions, Proposition 4.4 has a converse and the following proposition holds: the motion $g(t, q)$ of system $G_n$ is periodic if and only if it is $L^-$-stable and the kinetic energy $T[g(t, q)]$ is periodic.

Thus the recurrence of the kinetic energy of the negatively Lagrange-stable system of $n$ bodies fully defines the character of the recurrence of the motion of the system. Since the differential equations (1.1) of the motion of the system of $n$ bodies have, in particular, the energy integral from which we can obtain an explicit expression for the force function $U$ of system $G_n$, it follows that all arguments and discussions can be applied to the function $U$. Such an approach may be convenient when the recurrence of the motions is checked experimentally, since when the masses are known, the force function depends on the distance between the bodies of the system.

*Conclusions.* 1°. The isochronism, i.e. the mutual comparability with respect to time recurrence of the kinetic energy and motion of the system of $n$ bodies is the necessary condition for the Lagrange stability of motion of such a system.

2°. The motion of an $n$-body system is determined by a $6n$-dimensional vector function, and its kinetic energy by a scalar function depending on the last $3n$ components of the velocities of motion of the system. The isochronism of the $6n$-dimensional vector function $g(t,q)$ and scalar functkon $T[g(t, q)] = T(t)$ is characterized by the fact that the Lagrange stability is a special property of the motion of an $n$-body system. Since the Lagrange stability represents one of the possible forms of stability of the motion, it is possible for the kinetic energy to be minimal in some sense along the trajectory of the Lagrange-stable motion of an $n$-body system. In particular, this is the case for the kinetic energy of a recurrent, almost periodic and periodic motion of an $n$-body system. In the cases discussed above the kinetic energy is minimal in the Birkhof sense.

3°. Energy constructions have a long history in celestial mechanics. However, this is apparently the first time that the energy integral and its corollaries have been applied directly to the qualitative study of the motion of an $n$-body system in the form given here. We also note that the basic results remain valid for other forms of interaction between bodies, provided that they depend on the distance only.

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# THE SEPARATION OF MOTIONS IN SYSTEMS WITH RAPIDLY ROTATING PHASE*

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In different versions of the method of averaging, the motion is separated into rapid oscillations and a slow drift with an accuracy depending on the order of approximation. It is shown below that in analytic systems with rapidly rotating phase this separation can be achieved so that the error is exponentially small. The remaining small error is shown to be theoretically impossible to eliminate in any version of the averaging method. From the statement of exponentially exact separation of oscillations and drift it follows in particular that the time the adiabatic invariant is maintained in single-frequency Hamiltonian systems (such as a pendulum with a slowly varying frequency, a charged particle in a weakly inhomogeneous field, etc.) is exponentially large. This statement is also used to prove that the splitting of the separatrix that occurs in the neighbourhood of resonance close to an integrable Hamiltonian system with to degrees of freedom, is exponentially small.

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